Caltech – Ph106b – Winter 2002

Problem Set 3 *due 1-29-2002*

Look out! This is not an easy set, I think.

1. (Adiabatic invariants vs. parametric resonance) The theory of adiabatic invariants says that the action variable has its average time derivative $\langle \dot{I} \rangle$ bounded by an quantity of order $T\left[O(\dot{b}^2) + O(\ddot{b})\right]$. Study of parametric resonance shows that weak, slow driving of a harmonic oscillator can drive a parametric resonance. Is there a conflict here? This problem aims to show that there isn't, but that an adiabatic invariant does meaningfully constrain the Floquet exponent.

   (a) For the Mathieu equation, rescaled to look like this:

   $$\frac{d^2\theta}{dt^2} + \left(1 - \frac{2q}{a}\cos\frac{2}{\sqrt{a}}t\right)\theta = 0\,,$$

   there is a parametric resonance for $a = n^2$, n an integer. Say why large n qualifies as an adiabatic deformation, and express the bound implied by adiabatic invariance on ΔI over one period of the driving force. You should get something of the form

   $$|\Delta I| \leq \sum_{i=1}^{2} c_i q^{\beta_i} a^{\gamma_i}\,,$$

   where β_i and γ_i are exponents you should calculate, and the c_i are constants that you need not determine (they depend on I).

   (b) Re-express your answer from the previous part as a bound on the imaginary part of the Floquet exponent μ (see (10.15) for a definition). I claim that your bound will have an $O(q)$ term. Evaluate the coefficient on this term, numerically if necessary, for $n = 1, 2$. With standard Mathematica numerics, $n = 3$ was already giving me a result indistinguishable from 0. Thus it looked like the bound from part b) was indeed satisfied, but I couldn't be totally sure. If you can do better, or give an analytic treatment, great! But consider this part of the problem solved if you get the bound on $\mathrm{Im}\,\mu$ and the coefficients for $n = 1, 2$.

2. (Phase space orbits) Consider the Hamiltonian

   $$H = \frac{1}{2}p^2 + \frac{r}{2}q^2 + \frac{1}{4}q^4\,.$$

(a) Draw the phase space orbits for r positive, zero, and negative. Indicate in your figures the stable and unstable equilibrium points (if any). For $r < 0$, show the separatrix. Your figures need not be quantitatively precise.

(b) Write down the general formula for the period of the motion. For $r = 0$, how does the period scale with the total energy as the energy goes to zero? For $r < 0$, how does the period of the orbits diverge as E passes through 0?

3. (The standard map) H&F 11-3. Requires some numerics.

4. (Henon-Heiles potential) H&F 11-5. Requires some numerics. You may simply quote answers for part a), since we did this problem in Ph106a. Also show for part a) that the Hamiltonian is symmetric under 120° rotations in the x-y plane.