Math for physicists

Calculus and differential equations were developed simultaneously with classical mechanics, so they tend to reinforce one another. These notes are intended to remind you of some of the math tricks one often encounters in mechanics problems, as well as in many other areas of physics.

If you have math concerns, please come and talk to me—it's particularly important to us that an accidental lack of familiarity with inverse hyperbolic trigonometric functions not keep you from learning physics! And keep in mind Einstein's remark, "Do not worry about your difficulties in mathematics; I can assure you that mine are still greater."

- **First order differential equations**

  There are two types of first order equations which are easy to solve. The first is linear equations, which look like this:

$$\dot{x} + f(t)x = g(t)\,, \tag{1}$$

  where as usual $\dot{x} = dx/dt$. The method for solving this sort of equation is as follows: first introduce an "integrating factor"

$$F(t) = e^{\int^t ds\, f(s)}\,, \tag{2}$$

  and then multiply through by $F(t)$ to obtain the following equivalent forms:

$$\begin{aligned} \frac{d}{dt}\left[F(t)x(t)\right] &= F(t)g(t) \\ x(t) &= \frac{1}{F(t)}\int^t ds\, F(s)g(s)\,. \end{aligned} \tag{3}$$

  When we write $\int^t ds\, f(s)$ we mean the indefinite integral of f expressed as a function of t. An equivalent notation would be $\int_{t_0}^t ds\, f(s)$, and now the arbitrary lower limit t_0 is your integration constant. This integration constant in (2) cancels out in the end (check), so you have one meaningful integration constant total, coming from the integration in (3).

  The second type of equations which are easy to solve is separable equations. Most often they look like this:

$$\dot{x} = f(x)\,, \tag{4}$$

  but a more general form would be

$$\frac{dx}{dt} = f(x)g(t)\,. \tag{5}$$

That factored form on the right hand side is why we call these equations separable. The trick is to rearrange,

$$\frac{dx}{f(x)} = g(t)dt\,, \tag{6}$$

and then integrate,

$$\int \frac{dx}{f(x)} = \int dt\, g(t)\,. \tag{7}$$

If $g(t) = 1$, then the right hand side is just $t - t_0$, where t_0 is the integration constant. Then you can drop the integration constant from the left hand side. If $g(t) \neq 1$, you get some more or less complicated implicit equation between x and t, still with exactly one integration constant.

You can have differential equations for anything in terms of anything: x as a function of t was what we chose above, but you might sometimes find a convenient differential equation involving dx/dv where v is velocity. Just think to be sure that you have a well-defined problem, not a formal manipulation that might be nonsense.

- **Hyperbolic trig**

Hyperbolic trigonometric functions are just like ordinary trig functions, only more so. Ordinary trigonometric functions are most elegantly defined using de Moivre's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta\,, \tag{8}$$

and taking the real and complex parts leads to

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\,. \tag{9}$$

To get to hyperbolic functions, you "drop the i's and add the h's:"

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}\,. \tag{10}$$

One also defines tanh, coth, sech, and csch in the obvious way, *e.g.* $\tanh x = \sinh x/\cosh x$. Try graphing these functions to get a feel for what they look like. Every algebraic relation for trigonometric functions has an analog for hyperbolic functions, only with some funny signs on account of the i's that you dropped:

$$\begin{aligned}
&\cosh^2 x - \sinh^2 x = 1 \\
&\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sinh x = \cosh x \\
&\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y \\
&\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \\
&\int dx\ \tanh x = \log\cosh x + C \\
&\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C
\end{aligned} \tag{11}$$

and so on. The last integral you can get at gracefully through "hyperbolic trig-sub:" set $x = \cosh u$ and it falls right out. This happens to be the integral you need in order to get the shape of a minimal area surface of revolution (aka the catenary).

- **Integration by parts**
  This is a useful trick in many integrals. It goes like this:

$$\int dt\, g(t)\frac{df}{dt} = g(t)f(t) - \int dt\, \frac{dg}{dt} f(t)\,, \tag{12}$$

where $f(t)$ and $g(t)$ are arbitrary functions. This formula can be checked by differentiating with respect to t on both sides so that it becomes

$$g(t)\frac{df}{dt} = \frac{d}{dt}\left[g(t)f(t)\right] - \frac{dg}{dt}f(t)\,. \tag{13}$$

This is a rearrangement of the product rule for differentiation. An amusing example is the derivation of the integral of $\ln x$:

$$\begin{aligned}
\int dx\ \ln x = \int dx\,(\ln x)\frac{dx}{dx} &= x\ln x - \int dx\,\frac{d(\ln x)}{dx}x \\
&= x\ln x - \int dx\,\frac{1}{x}x = x\ln x - x + C\,.
\end{aligned} \tag{14}$$

- **Power series**
  Almost every function that has a name, and certainly every function which solves a simple differential equation, admits a power series expansion around all but a discrete set of points. That is,

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n\,, \tag{15}$$

where $f^{(n)}$ is the n'th derivative of f and x_0 can be chosen arbitrarily. A physicist seldom needs more than the first few terms. Particularly useful examples are

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + \ldots \\
\sin x &= x - \frac{x^3}{6} + \frac{x^5}{120} + \ldots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \ldots \\
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \ldots \\
\ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots .
\end{aligned} \tag{16}$$

At the special points where some functions don't admit a power series expansion, there's usually some variant of a power expansion which works, namely

$$\begin{aligned}
&\text{branch cut:} && f(x) = (x-x_0)^\gamma \sum_{n=0}^{\infty} c_n (x-x_0)^n \\
&\text{simple pole:} && f(x) = \frac{c_{-1}}{x-x_0} + \sum_{n=0}^{\infty} c_n (x-x_0)^n \\
&\text{multiple pole:} && f(x) = \sum_{n=-k} c_n (x-x_0)^n \qquad \text{for some integer } k > 0 .
\end{aligned} \tag{17}$$

Actually the second two forms are special cases of the first form with γ an integer, but it helps to distinguish the cases. You can work out the c_n by first multiplying left and right by $(x-x_0)^{-\gamma}$. Examples:

$$\begin{aligned}
\cot x &= \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + \ldots \\
\sqrt{\cot x} &= x^{-1/2} \left(1 - \frac{x^2}{6} - \frac{x^4}{40} + \ldots \right) .
\end{aligned} \tag{18}$$

The second one can be worked out by first Taylor expanding $\sqrt{x \cot x}$.

To extract a limit of a complicated function, it often helps to make a series expansion of its components. For instance

$$\lim_{t\to 0} \frac{\sin t \tanh t - t^2}{(e^t - 1)^4} = \lim_{t \to 0} \frac{(t - t^3/6)(t - t^3/3) - t^2}{t^4} = -\frac{1}{2} \tag{19}$$

Try that one with L'Hospital's rule!

Finally, to extract approximate behavior of $f(x)$ for large x, you can try to write a power series for $f(1/y)$ around $y = 0$.

- **All those derivatives!**

  Physicists have lots of different ways of writing derivatives. For us the most common are $\dot{x}(t) = \frac{dx}{dt}$ and $y'(x) = \frac{dy}{dx}$. When considering functions of several variables, particular care is needed in specifying how to take a derivative. Suppose you have a function $f(x, y, t)$, say

$$f(x,y,t) = A \sin \frac{2\pi}{L}(x + y - vt)\,. \tag{20}$$

  This function describes a wave traveling up and to the right. Think of it as a wave on the ocean. Partial derivatives are the simplest things to think about: for instance, $\frac{\partial f}{\partial x}$ is the slope of the wave in the $\hat{x}$ direction at a specified position and time, while $\frac{\partial f}{\partial t}$ is the rate at which f is rising at a fixed point (x, y). The essential thing is that we hold x and y fixed when computing $\frac{\partial f}{\partial t}$. Thus

$$\frac{\partial f}{\partial t} = \frac{2\pi v A}{L} \cos \frac{2\pi}{L}(x + y - vt)\,. \tag{21}$$

  Now, what is the difference between $\frac{\partial f}{\partial t}$ and $\frac{df}{dt}$? Well, often it happens that we would consider x and y to be definite functions of t. For instance, in our ocean analogy, $(x(t), y(t))$ could be the position of a surfer at time t. Then the surfer's height above sea level at time t is $f(x(t), y(t), t)$. We would *still* define $\frac{\partial f}{\partial t}$ as in (21), as the derivative of $f(x, y, t)$ with respect to t, holding x and y fixed. But we define the total $\frac{df}{dt}$ as the change in height over time of the surfer. It's different from $\frac{\partial f}{\partial t}$ because the surfer is moving. By the chain rule,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}\,. \tag{22}$$

  For instance, if the surfer travels with the wave, so that $(x(t), y(t)) = (vt/2, vt/2)$, then $f(x(t), y(t), t) = 0$ for all t, so $\frac{df}{dt} = 0$ even though $\frac{\partial f}{\partial t}$ is nonzero (it's still given by (21)). If you want, you can trace through the terms on the right hand side of (22) and see them cancel each other out.

  When we write Lagrange's equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}\,, \tag{23}$$

  both types of derivatives are involved. Given the lagrangian $L(q, \dot{q}, t)$, the partial derivative $\frac{\partial L}{\partial q}$ means we hold $\dot{q}$ fixed and differentiate with respect to q. The partial derivative $\frac{\partial L}{\partial \dot{q}}$ means we hold q fixed and differentiate with respect to $\dot{q}$. It's OK to treat q and $\dot{q}$ as independent variables in this setting: that's what partial derivatives are about. But when you apply d/dt on the left hand side of (23), it's a total derivative.

- **Matrices**

  Any number of good expositions can be found; Marion and Thornton's *Classical Dynamics* has an elementary one. Vectors, dot products and cross products of vectors, matrices, products of matrices, inverses of matrices, transposes of matrices, and determinants are worth re-acquainting yourselves with if they're not familiar. We will also encounter eigenvalues and eigenvectors. A matrix $\mathbf{M}$ has an eigenvalue λ and associated eigenvector $\vec{v} \neq 0$ if

$$\mathbf{M}\vec{v} = \lambda\vec{v}\,. \tag{24}$$

  For small matrices, the method of choice to determine the eigenvalues is to note that since $\mathbf{M} - \lambda\mathbf{1}$ annihilates a non-zero vector (namely $\vec{v}$), it must have non-maximal rank, hence zero determinant:

$$\det(\mathbf{M} - \lambda\mathbf{1}) = 0\,. \tag{25}$$

  This "characteristic equation" is a polynomial equation for lambda whose degree is equal to the size of the matrix. Once you have all the roots, you can go back to (24) to find $\vec{v}$ for each root.

  A notation which physicists love is the Kronecker delta function, δ_{ij}, which is defined like this:

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j\,. \end{cases} \tag{26}$$

  If i and j run from 1 to n, then $[\delta_{ij}]$ is the $n \times n$ identity matrix (denoted above by $\mathbf{1}$). Note that $\sum_j \delta_{ij} v_j = v_i$: this is the obvious identity $\mathbf{1}\vec{v} = \vec{v}$ written out in components.

- **Dirty tricks**

  We highly encourage you to become proficient in the manipulation of simple differential equations, matrices, hyperbolic functions, integrals, etc. Not only will it help you solve the problems efficiently; it will also improve your grasp of calculus, which no one can know too well. Furthermore, as Elliott Lieb (mathematical physicist extraordinaire) intoned to me ten years ago, "integration is good for the soul."

  Now that the preaching is over, let's get down to the dirty tricks that can save you time. The first is Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products.* This tome is not exactly bedtime reading, but it contains an incredible array of integrals—I still haven't encountered a single do-able integral that they hadn't done (whereas I can fox Mathematica any day of the week). If the usual tricks (u-substitution, trig-sub, hyperbolic-trig-sub, and integration by parts) seem to be getting you nowhere, or seem to work but give very long formulas, crack open this book and have a look.

The second is Mathematica. The commands `Integrate`, `Eigensystem`, `DSolve`, `Series`, and `Simplify` are fairly powerful. Maple has analogous commands but I don't know them since I don't use Maple. Matlab seems somewhat more limited when it comes to symbolic manipulation. To use any of these programs you have to invest a little time figuring out the syntax. In Mathematica, you can say, for instance, `??Series`, to figure out the syntax for the `Series` command. Mathematica is a very practical tool. Try it, it's worth it!